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Synchronization of randomly driven nonlinear oscillators

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When nonlinear oscillators with stable limit cycles are subject to periodic forces, these oscillators may become entrained or mode locked to the driving force. Remarkably, a similar phenomenon occurs when the nonlinear oscillators are driven by a random force. In particular, when nonlinear oscillators with different initial conditions are strongly driven with the same random force, their fluctuating behavior may reliably converge to an identical, synchronized response. Analytical estimates are derived for the conditions, rates, and structural stability for synchronization of a broad class of randomly driven nonlinear oscillators, which suggest different experimental procedures for assessing the nonlinear response of biological, chemical, and physical oscillators to fluctuating inputs. [S1063-651X(98)50612-5]

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Mathematical models of nonlinear oscillators describe a wide variety of physical and biological phenomena that exhibit self-sustained oscillatory behavior, from the the van der Pol (vdP) equations for nonlinear electrical circuits, to the Hodgkin-Huxley model for spiking neurons, to simple phase models for the flashing of fireflies [1]. When strongly driven by forces that are periodic in time, these oscillators may become entrained to oscillate at the same frequency as the driving force. This phenomenon of ''mode locking'' or ''phase locking'' is epitomized by Huygens's ancient observation that pendulum clocks hung from the same beam tend to synchronize due to the common periodic vibrations of the beam [2]. Although there is a vast literature [3–5] on this ubiquitous phenomenon, there are few analyical results. Qualitatively, mode locking occurs when the periodic perturbation is sufficiently strong and the frequency of the perturbation is sufficiently close to the unperturbed frequency (or to a rational multiple) of the nonlinear oscillator. One important consequence of this effect is that a common periodic perturbation can force two nonlinear oscillators, which start out with different initial conditions, to converge to identical, synchronized states.

Remarkably, this synchronization also occurs when nonlinear oscillators are driven by random forces. In this case it is not clear what the oscillator is mode locking to, since the driven oscillator also exhibits random fluctuations. Nevertheless, random forcing applied to two identical nonlinear oscillators with different initial conditions may reliably lead to asymptotically stable, synchronized states. This phenomenon is graphically illustrated in Fig. 1, which displays the evolution of two van der Pol oscillators [3,5],

$$\frac{d^2x}{dt^2}=\epsilon(1-x^2)\frac{dx}{dt}-\Omega^2x+F(t), \tag{1}$$

with different initial conditions, driven by a common random force $F(t)$.

Recently, there has been considerable interest in the synchronization of chaotically driven nonlinear systems [6,5]. However, chaos is not necessary for the realization of this striking effect. The chaotically driven systems are special cases of the more general phenomenon of the possibility of asymptotically stable [7] solutions for aperiodically driven nonlinear systems. A simple example of the synchronization of randomly driven nonlinear oscillators was believed to be first observed by Pikovskii in 1984 [8].

This synchronization of aperiodically or randomly driven nonlinear oscillators is structurally stable, which means that approximate synchronization is realized even in the presence of small variations (or errors) in the parameters of the systems and in small levels of additional noise in the driving signal. This structural stability is essential for practial applications in secure communication, control theory, and synchronization of biological oscillators, since real systems will inevitably have small variations and be subject to background noise [6].

For example, nonlinear oscillators used to model the spiking voltage dynamics of neurons, such as the Fitzhugh-Nagumo, Morris-Lecar, and Hodgkin-Huxley equations [9], are all found to exhibit asymptotically stable, synchronized behavior when strongly driven by randomly fluctuating currents [10]. This effect provides a dynamical mechanism for the remarkable reliability of the spike timing recently observed [11] in the response of neocortical neurons to fluctuating input currents that resemble real synaptically generated currents. The same levels of spike timing reliability are achieved in the different mathematical models for neuronal

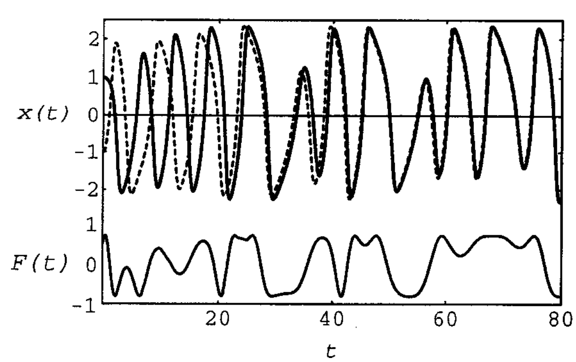

FIG. 1. Time-evolution of $x(t)$ is plotted for two randomly driven vdP oscillators with $\Omega=1$ and $\epsilon=1$. The driving force $F(t)=0.8\sin\phi(t)$ with a Gaussian random phase $\phi(t)$ is shown in the lower curve. Starting from two different initial conditions, $x(0)=1$ (solid line) and $\overline{x}(0)=-1$ (dashed line), the solutions converge to indistinguishable curves by $t=50$.

oscillations, even with small changes in initial conditions, cell parameters, and driving currents [10]. This reliability of spike timing is essential if single neurons are expected to faithfully encode temporal information in the timing of successive spikes [12].

Although damped, driven linear oscillators always lead to asymptotically stable solutions [7], the asymptotic stability of driven nonlinear oscillators is very much dependent on the parameters and properties of the nonlinear system, and of the drive. Formally, this stability can be established by demonstrating that all of the Lyapunov exponents are negative or by constructing an appropriate Lyapunov function [7,5]. However, in general, the Lyapunov exponents can only be calculated numerically for specific systems and the rigorous Lyapunov functions can only be constructed in special cases.

To provide a theoretical foundation for the general phenomenon of the synchronization of nonlinear oscillators by aperiodic inputs, this paper will focus on nonlinear oscillators that are well approximated by simple phase models [13]. General analytical expressions are derived to provide quantitative estimates of the conditions for observing the synchronization for this broad class of randomly driven nonlinear oscillators as a function of the fluctuation amplitudes and frequency spectra along with estimates for the rates of convergence to the synchronized solutions and the requirements for structural stability. These results reveal that the synchronization of randomly driven oscillators is closely related to the periodically driven case, which suggests different experimental procedures for assessing the nonlinear response of biological, chemical, and physical oscillators to fluctuating inputs. Finally, both the analytical and empirical analyses are used to successsfully predict the synchronization of randomly driven vdP oscillators (such as those displayed in Fig. 1).

First, consider two nonlinear oscillators with stable limit cycles, $x(t)\equiv A(t)\sin\theta(t)$ and $y(t)\equiv B(t)\sin\phi(t)$, characterized by their phases $\theta(t)$ and $\phi(t)$. Very generally, the interaction of two (weakly coupled) nonlinear oscillators can be expressed in terms of the phase differences alone [13]. For example, the evolution of the x oscillator may be approximately described by the simple phase model [14]

$$\frac{d\theta}{dt}=\Omega-b\sin[\theta(t)-\phi(t)]. \tag{2}$$

If we assume unidirectional coupling from y to x, and allow the time-dependent phase $\phi(t)$ in Eq. (2) to be an arbitrary function of time, then Eq. (2) can be treated as a simple, ''toy'' model for studying the response of a nonlinear oscillator to random phase noise. The analysis is further simplified by defining $\eta(t)=\theta(t)-\phi(t)$ and rewriting Eq. (2) as

$$\frac{d\eta}{dt}=[\Omega-\omega(t)]-b\sin\eta(t), \tag{3}$$

where $\omega(t)\equiv d\phi(t)/dt$. Then a variety of different responses are possible depending on the magnitude of the coupling b and the time dependence of the drive frequency $\omega(t)$.

Case 1. The phase model results for constant frequency drive $\phi(t)=\omega t$ are well known [5]. When ω lies within the stable zone close to the natural frequency Ω,

$$\left|\frac{\Omega-\omega}{b}\right|<1, \tag{4}$$

the driven oscillator $x(t)$ will mode lock to the drive. Starting from any initial phase, the long-time solution to Eq. (2) converges to $\theta(t)=\omega t+\eta$, which has the same frequency as the periodic drive but is shifted by a constant phase $\eta=\arcsin[(\Omega-\omega)/b]$. The mode-locked dynamics of the driven oscillator $x(t)$ is asymptotically stable [7], and the Lyapunov exponent that determines the exponential rate of convergence, γ, can be calculated by a simple perturbation analysis of Eq. (3) [5],

$$\gamma=b\cos\eta=\sqrt{b^2-(\Omega-\omega)^2}. \tag{5}$$

One straightforward but important consequence of this asymptotic stability is that the long-time behavior of two driven oscillators $x(t)$ and $\overline{x}(t)$, with different initial phases θ_0 and $\overline{\theta}_0$, will always *synchronize* (i.e., $|\theta(t)-\overline{\theta}(t)|\sim e^{-\gamma t}$) as the individual solutions converge (at the exponential rate γ) toward the common asymptotic solution [15].

Another important feature of the synchronization of periodically driven nonlinear oscillators is that this behavior is structurally stable. If two different oscillators, x and $\overline{x}$, with slightly different natural frequencies $\Omega-\overline{\Omega}=\delta\Omega$ or coupling strengths $b-\overline{b}=\delta b$ are driven by the same periodic drive with frequency ω inside the respective stable zones defined by Eq. (4), then both oscillators will still be entrained to the periodic drive and will appear to be approximately synchronized with a small, constant phase difference $\delta\eta=\eta-\overline{\eta}\approx[\delta\Omega-(\Omega-\omega)(\delta b/b)]/\gamma$.

On the other hand, if the driving frequency ω is too far away from the natural frequency Ω or if the coupling b is too weak [so that Eq. (4) is not satisfied], then there is no asymptotic convergence to a common, mode-locked solution, and there is no synchronization. The evolution of $x(t)$ is quasiperiodic and different initial conditions lead to distinct long-time solutions.

Case 2. For time-dependent drive frequencies $\omega(t)$, nearby solutions will tend to converge when the frequency lies within the stable zone,

$$\left|\frac{\Omega-\omega(t)}{b}\right|<1. \tag{6}$$

In particular, if the frequency of the periodic drive $\omega(t)$ varies slowly through the stable zone, then the oscillator angle $\theta(t)$ will still try to track the angle of the drive $\phi(t)$ with a time-dependent phase shift $\eta(t)\approx\arcsin\{[\Omega-\omega(t)]/b\}$, and nearby solutions will converge exponentially at an approximate instantaneous rate,

$$\gamma(t)\approx\sqrt{b^2-[\Omega-\omega(t)]^2}. \tag{7}$$

A more detailed analysis [16] of the phase model with time-dependent drive frequency indictates that a sufficient condition for the validity of these adiabatic approximations is provided by the requirement that the transit time through the stable zone be long compared with the convergence time or, equivalently, that ω and the rate of change of ω be small compared with the rate of synchronization,

$$\frac{d\omega(t)}{dt}\bigg/\gamma(t)^2<1. \tag{8}$$

When conditions (6) and (8) are met, two nearby solutions will converge toward synchronization as they approach the common asymptotic solution during the transit through the stable zone. As in the constant frequency case, this synchronization is structurally stable in the sense that the evolution of two different oscillators $x(t)$ and $\overline{x(t)}$, with slightly different values for the natural frequencies $\delta\Omega$ or coupling strengths δb, will approximately synchronize with a small (bounded) phase difference $\delta\eta=\eta-\overline{\eta}\approx\{\delta\Omega-[\Omega-\omega(t)](\delta b/b)\}/\gamma(t)$.

When $\omega(t)$ leaves the stable zone, $\theta(t)$ will no longer be constrained to closely track the driving angle and two driven oscillators will tend to desynchronize. Nevertheless, synchrony will be restored when the drive frequency enters the stable zone again as long as the transit time through the stable zone is long compared with the convergence time. Then the mean rate of synchronization Γ can be estimated by taking the time average of the instantaneous convergence rate

$$\Gamma=\frac{1}{T}\int_0^T\gamma(t)dt, \tag{9}$$

using Eq. (7) when the drive frequency is in the stable zone and $\gamma(t)=0$ otherwise.

Case 3. For randomly varying drive frequencies the determination of Γ can be simplified by replacing the time average in Eq. (9) with an ensemble average. For example, if $\phi(t)=\phi_r(t)$ is a Gaussian random process [17] with a power spectrum $S(\nu)$ and variance $\sigma\equiv(1/2\pi)\int_{-\infty}^{\infty}S(\nu)d\nu$, then the probability distribution $P(\omega_r)$ for the derivative of the random phase $\omega_r\equiv d\phi_r/dt$ will also be Gaussian [17] with variance $\sigma_\omega=(1/2\pi)\int_{-\infty}^{\infty}\nu^2S(\nu)d\nu$. In this case the adiabaticity parameter in Eq. (8) can be estimated using the average ratio $R_A=\sqrt{\sigma_{\omega'}}/b^2$, where $\sigma_{\omega'}\equiv(1/2\pi)\int_{-\infty}^{\infty}\nu^4S(\nu)d\nu$ is the variance of $d\omega_r(t)/dt$ and b is the maximum convergence rate at the center of the stable zone. Then, when R_A is small, driven oscillators may be expected to synchronize at a mean convergence rate,

$$\Gamma=\int_{\Omega-b}^{\Omega+b}\sqrt{b^2-(\Omega-\omega_r)^2}P(\omega_r)d\omega_r. \tag{10}$$

In particular, if ω_r has zero mean and the stable zone is narrow, $b\ll\Omega$, then Eq. (10) can be further approximated by assuming that $P(\omega_r)=(1/\sqrt{2\pi\sigma_\omega})e^{-\omega_r^2/2\sigma_\omega}$ does not change much over the stable zone,

$$\overline{\Gamma}\approx\frac{b^2\pi}{2}\frac{1}{\sqrt{2\pi\sigma_\omega}}e^{-\Omega^2/2\sigma_\omega}. \tag{11}$$

Figure 2 shows that the predictions of Eq. (11) are in good agreement with numerically calculated synchronization rates (large dots) for driven phase oscillators with $b=0.5$, $\Omega=1$, and frequency distributions with $\sigma_\omega=0.25$ to 4 and $R_A=0.5$ to 2. The breakdown of the adiabatic theory is also illustrated in Fig. 2 by the significantly reduced synchronization rates (small dots) for rapidly fluctuating driving forces with broad frequency distributions and large $R_A=5$–20.

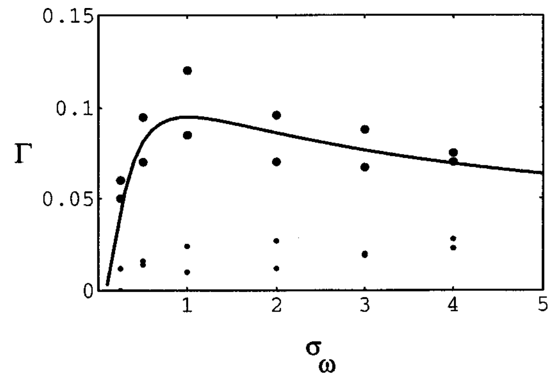

FIG. 2. Predicted synchronization rates Γ for randomly driven phase oscillators with $\Omega=1$ and $b=0.5$ (solid curve) are compared with the results of numerical simultations as functions of σ_ω for small values of the adiabaticity parameter (large dots) with R_A between 0.5 and 2 (plotted from left to right), and for large $R_A=5$–20 (small dots). The synchronization rates in the numerical simulations are calculated by measuring the average exponential rate of convergence of the solutions for two identical driven oscillators with different initial conditions. The different dots plotted for each σ_ω correspond to different realizations of the random drive.

Case 4. The classic van der Pol oscillator [Eq. (1)] is a harmonic oscillator with nonlinear damping that exhibits a stable limit cycle [5]. If the solution is expressed in oscillatory form $x(t)\equiv a(t)\sin\theta(t)$, then for small ϵ the van der Pol equations can be approximated by equations for the amplitude $a(t)$ and phase $\theta(t)$, derived by neglecting rapidly oscillating terms [5,4],

$$\frac{da}{dt}\approx\frac{\epsilon a}{2}(1-a^2/4), \tag{12}$$

$$\frac{d\theta}{dt}\approx\Omega. \tag{13}$$

In this case every solution converges to the limit cycle $x(t)\approx2\sin[\theta(t)]$ where $\theta(t)=\Omega t+\theta(0)$.

If the van der Pol oscillator $x(t)$ is driven with a force $F(t)=B\sin\phi(t)$, then for small ϵ the equations for the oscillatory form [4] of the solution can again be approximated by neglecting rapidly oscillating terms,

$$\frac{d\theta}{dt}\approx\Omega-\frac{B}{4}[\cos(\theta-\phi)-\cos(\theta+\phi)]. \tag{14}$$

In particular, when the driving term has a constant frequency ω, Eq. (14) can be approximated further by neglecting the rapidly oscillating term $\cos(\theta+\omega t)$ to arrive at the simple phase model

$$\frac{d\theta}{dt}\approx\Omega-\frac{B}{4}\cos[\theta(t)-\omega t], \tag{15}$$

and synchronization of two vdP oscillators may be expected when $|\Omega-\omega|<B/4$ [18].

Using the correspondence between $b\equiv B/4$, the previous results for the phase model provide useful estimates for the average rate of synchronization for weakly nonlinear ($\epsilon\ll1$) van der Pol oscillators with random driving. For example, in Fig. 1 two van der Pol oscillators with $\epsilon=1$ and $\Omega=1$ are synchronized by a random force with $B=0.8$ and $\sigma_\omega=1$ at an average exponential rate of $\Gamma\simeq 0.04$. This compares well with the predicted rate $\Gamma=0.03$, derived by doubling the rate given by Eq. (11) with $b=0.2$ because of the dual contributions from positive and negative frequencies in Eq. (14).

Because the van der Pol oscillator admits additional stable zones near harmonics and subharmonics of the natural frequency of the undriven oscillator, the phase model tends to underestimate the synchronization rate for variable frequency drives. However, the success of this adiabatic analysis suggests a practical procedure for assessing the asymptotic stability for strongly nonlinear oscillators when the reduction to the simple phase model breaks down. In these cases the synchronization rate can still be estimated for slowly varying frequency drives by first measuring the convergence rates for an array of constant drive frequencies and then using these results to average the instantaneous convergence rate over the randomly fluctuating drive. Figure 3 demonstrates the utility of this general program for quantitatively characterizing the synchronization of randomly driven nonlinear oscillators, using the example of a strongly nonlinear van der Pol oscillator with $\epsilon=4$ subject to strong random forces with $B=1$ and 2.

A wide variety of nonlinear oscillators are found to exhibit reliable synchronization when driven by either periodic or aperiodic inputs. This paper extends results for constant frequency drives to slowly fluctuating drives and provides analytical estimates for the conditions, rates, and structural stability for the synchronization of randomly driven phase oscillators. These results suggest a different way of thinking about the response of nonlinear oscillators to aperiodic inputs by demonstrating the close relationship to the periodically driven case. Specifically, the reliable response of a nonlinear oscillator to a fluctuating input may be more dependent on the frequency spectrum of the input than on the overall amplitude. Finally, this work suggests a practical procedure for theoretically and experimentally evaluating the conditions and rates of synchronization for more realistic nonlinear oscillators and aperiodic driving forces by first measuring the response to different periodic drives and then averaging these results over the spectrum of the random drive.

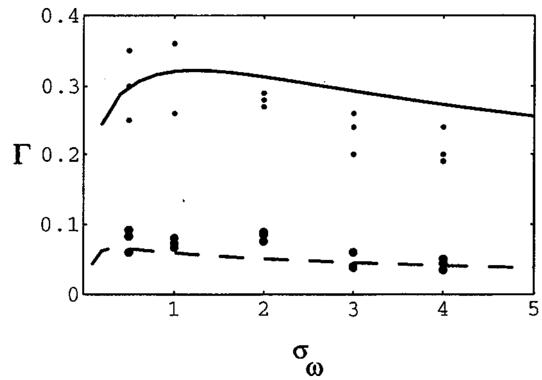

FIG. 3. The predicted synchronization rates Γ for randomly driven van der Pol oscillators with $\epsilon=4$, evaluated using the numerical results for periodic driving alone (curves), are compared with the results of direct numerical simulations (dots) for $B=2$ (solid curve and small dots) and $B=1$ (dashed curve and large dots) as functions of σ_ω. Remarkably, the adiabatic estimates show good agreement even though the approximate adiabaticity parameter for the random phase drive varies from $R_A=1$ to 16 in these simulations.

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